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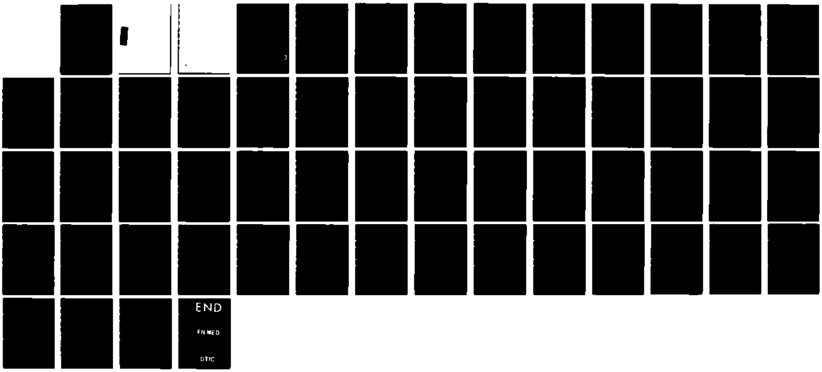

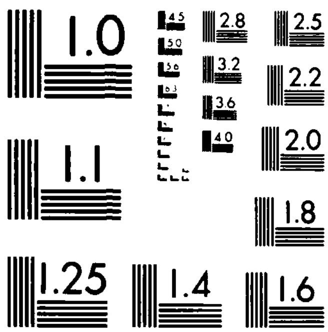

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SCATTERING FROM CONDUCTING BODIES OF REVOLUTION: BEHAVIOR OF THE INTEGRAL EQUATIONS NEAR SINGULAR POINTS OF THEIR KERNELS

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ABSTRACT

The basic formulation of scattering problems in terms of integral equations is examined, for the special case of perfectly conducting bodies of revolution. In particular, the singularities of the integrals which arise in this context are studied, in relation to the transition to a numerical solution by means of the method of moments. For both H-field and E-field equations, it is found that finite matrix elements can be deduced in a way that is uniquely determined by the integrals themselves. No ad hoc procedures are required to secure convergence, but one such procedure, which is commonly used, is shown to be capable of giving accurate results for those integrals which tend to diverge logarithmically. In the H-field solution, other integrals also arise, which yield finite terms not normally included in the theory, when the necessary limiting procedures are carried out. These terms can play a significant role in the case of a body whose profile contains a corner.

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INTRODUCTION

The formulation of scattering problems in terms of integral equations is a well-established procedure, and the application of numerical techniques to the solution of these equations has spawned a large and varied literature. The integrals in these formulations are generally improper, in the sense that the integrands, being essentially Green's functions, have integrable singularities. The integrals must therefore be treated with some care, particularly when the transition is made to a set of linear equations by means of the method of moments. The diagonal elements of the resulting matrices are directly related to regions of integration where these singularities manifest themselves. Many analyses have been carried out in which divergent integrals are replaced by finite expressions or convergent integrals in a somewhat ad hoc way, the procedure being justified by physical reasoning. It is always recognized that these divergences arise because of oversimplifications in the application of the numerical procedure, such as the use of delta-function expansion or test functions in the method of moments. It is assumed that errors due to the approximations used will turn out to be small, and indeed this is so when the results are compared to known cases, such as the sphere. However, one never knows, a priori, that the calculations will be as accurate when the technique is applied to a body of radically different

shape. It is generally accepted that the method of moments procedure itself will converge more rapidly (as a function of the number of points used to represent the body) if its matrix elements are computed with increased precision, and it may even be possible to gain assurance that the answers to which the method converges are in fact the correct ones. It therefore seems desirable to formulate the problem in a way that represents the integrals accurately during the transition to a system of linear equations.

In this note the singular kernels are examined in the special case of scattering by conducting bodies of revolution, and it is found that finite diagonal matrix elements can be deduced in a way that is uniquely determined by the integrals themselves. Although the expressions found are slightly different than those customarily used in the literature, it is not expected that the numerical results which follow from their application will be very different, with the possible exception of bodies with corners. In this case the present evaluations (for the H-field solution) contain terms not generally used and which are directly related to the abrupt change of the tangent direction of the profile of the body at a corner.

The analysis is carried out here in terms of the H-field equation, since it is an integral equation of the second kind with no derivatives operating on the unknown function. For these

reasons, it has the better chance of numerical stability, at least away from resonant frequencies of the corresponding internal problem. In the usual treatment of the E-field problem, derivatives of the expansion and test functions appear, which are approximated by difference operations. This adds another element of inaccuracy to the analysis, producing errors which are difficult to assess in advance. Here also, divergent integrals occur, and these can easily be treated by the methods given here. The E-field solution is discussed in the final section of this report.

1. FORMULATION of the PROBLEM.

Let S be a closed surface, covering a perfectly conducting body, and let $H_O(r)$ be the magnetic field which would be produced by a system of sources in the absence of this body. With the body present, the total field produced by this same system of sources satisfies the equation:

$$H(r) = H_o(r) + \int_S K(r') \times \nabla' g(r,r')\, dS'$$

where r' is the variable of integration on the surface, ∇' is the gradient relative to r', dS' is the surface element, $K = \underline{n} \times H$, g(r,r') is the scalar Green's function:

$$g(r,r') = \frac{e^{ikR}}{4\pi R} \quad , \quad R = \left| r - r' \right|$$

and k is the wavenumber. The suppressed time dependence is of the form $e^{-i\omega t}$, and only unit vectors, indicated by an underline, are specifically identified as vectors in this report. The field point, r, is exterior to the surface in the above equation and K depends on the unknown values of tangential H on the surface. By letting r approach the surface, and taking the limiting value of the integral, we obtain the Maue[1] equation for K:

$$K_o(r) = \frac{1}{2} K(r) - \underline{n} \times \int_S K(r') \times \nabla' g(r,r')\, dS'$$

where $K_o = \underline{n} \times H_o$. In this integrand, r is on S, and the surface integral is actually convergent. The formulation of the scattering problem in terms of this equation for a body of revolution is well known,[2] but we give the basic steps here in order to make the development reasonably self-contained.

We define the function

$$G(R) = (1-ikR)\,\frac{g(R)}{R^2}$$

so that

$$\nabla' g(r,r') = (r-r')\,G(R)$$

and the Maue equation becomes

$$K_o(r) = \frac{1}{2} K(r) + \int_S \underline{n} \times \left[(r-r') \times K(r')\right] G(R)\, dS'.$$

We let $\underline{p}$ represent either of two orthogonal tangential unit vectors at r, $\underline{p}'$ being one of the unit vectors at r', and then write the integral equation in component form. Indicating vector components by superscripts, we have:

$$K_o^p(r) = \frac{1}{2} K^p(r) + \sum_{p'} \int_S M^{pp'}(r,r')\, K^p(r')\, dS'$$

where $\sum$ sums over the two values of p', and the dyadic kernel has components

$$M^{pp'}(r,r') = \underline{p} \cdot \underline{n} \times \left[(r-r') \times \underline{p}'\right] G(R) .$$

Next we specialize the surface to that of a body of revolution, taking the axis of symmetry as our z-axis, and introduce a coordinate system for points and vectors on S. The three-dimensional cylindrical (ρ,ϕ,z) system will also be employed, where ρ is the distance from the z-axis and ϕ is the azimuthal angle. We suppose that the body half-profile is defined by the relation

$$\rho = \rho(z)$$

and introduce the arc-length, t, along the body by means of its differential element:

$$dt = (dz^2 + d\rho^2)^{1/2}.$$

The derivative, $d\rho/dz$, is related to β, the angle which the tangent to the profile makes with the z-axis, by the equation:

$$\frac{d\rho}{dz} = \tan \beta$$

and in terms of β we have

$$dz = \cos\beta \, dt \quad , d\rho = \sin\beta \, dt .$$

For our purpose it is useful to think of t as the independent variable, measured from a reference point (z_o,ρ_o) on the body, and to express β as a function of t.

Then z and ρ are obtained as integrals:

$$z = z_o + \int_{t_o}^{t} \cos\beta(s)\, ds$$

$$\rho = \rho_o + \int_{t_o}^{t} \sin\beta(s)\, ds.$$

These equations will play a basic role in the analysis of the singular portions of the integral equation for K. The coordinates of a surface point are now t and ϕ, and the surface element itself is $dS = \rho dt d\phi$.

Let $\underline{t}$ be a tangential unit vector in the direction of increasing t, and let $\underline{\phi}$ be a similar vector, in the (ρ,ϕ)-plane, in the direction of increasing ϕ. Together with the normal unit vector, $\underline{n}$, these form a local orthogonal system. We assume that the postive t-direction is defined such that

$$\underline{n} = \underline{\phi} \times \underline{t} \quad .$$

It is useful to have the notation

$$\underline{q}(\underline{p}) = \underline{p} \times \underline{n}$$

so that

$$\underline{q}(\underline{t}) = \underline{\phi} \quad , \; \underline{q}(\underline{\phi}) = -\underline{t} \; .$$

Returning to the dyadic, M, we have

$$M^{pp'}(r,r') = \left[(r-r') \times \underline{p}'\right] \cdot (\underline{p} \times \underline{n})\, G(R)$$

$$= \underline{q} \cdot [(r-r') \times \underline{p}'] \, G(R)$$

$$= \{(\underline{q} \times r) \cdot \underline{p}' + \underline{q} \cdot (\underline{p}' \times r')\} \, G(R)$$

The dot products can be conveniently evaluated in the cylindrical coordinate system, and this is carried out in Appendix 1 for the more general case in which r is off the surface a small distance in the normal direction. When specialized to a surface point for r, we obtain the dyadic elements in the form derived by Harrington[2]:

$$M^{tt}(r,r') = \{[(\rho'-\rho)\cos\beta' - (z'-z)\sin\beta'] \cos\psi$$

$$- 2\,\rho \cos\beta' \sin^2(\psi/2)\} \, G(R)$$

$$M^{t\phi}(r,r') = (z'-z) \sin\psi \, G(R)$$

$$M^{\phi t}(r,r') = [\rho'\sin\beta \cos\beta' - \rho \cos\beta \sin\beta' - (z'-z) \sin\beta \sin\beta'] \cdot$$

$$\sin\psi \, G(R)$$

$$M^{\phi\phi}(r,r') = \{[(\rho'-\rho)\cos\beta - (z'-z)\sin\beta] \cos\psi$$

$$+ 2\,\rho' \cos\beta \sin^2(\psi/2)\} \, G(R)$$

In these formulas, $\psi = \phi' - \phi$.

The dyadic depends on the azimuthal angles only through ψ, as a consequence of the cylindrical symmetry, and the next step is the modal expansion. Substituting

$$K^p(r) = \sum_m K^p(t;m) \, e^{im\phi}$$

and defining

$$K_o^p(t;m) = \frac{1}{2\pi} \int_0^{2\pi} K_o^p\,(t,\phi)\; e^{-im\phi} d\phi$$

and

$$L^{pp'}(t,t';m) = \int_0^{2\pi} M^{pp'}(r,r')\; e^{im\psi}\; d\psi$$

we obtain the uncoupled modal integral equations

$$K_o^p(t;m) = \frac{1}{2}\, K^p(t;m) + \sum_{p'} \int L^{pp'}(t,t';m)\; K^{p'}(t';m)\rho' dt'.$$

In R, the argument of G, the introduction of cylindrical coordinates yields

$$R = \left| r-r' \right| = \left[(\rho'-\rho)^2 + (z'-z)^2 + 4\rho'\rho \sin^2(\psi/2) \right]^{1/2} .$$

Since R is an even function of ψ, the ψ-integrals involved in the components of L reduce to the following:

$$T_1(t,t':m) = 2 \int_0^{\pi} G(R)\; \cos\psi\; \cos(m\psi)\; d\psi$$

$$T_2(t,t';m) = 2 \int_0^{\pi} G(R)\; \sin^2(\psi/2)\; \cos(m\psi)\; d\psi.$$

$$T_3(t,t';m) = 2\; i \int_0^{\pi} G(R)\; \sin\psi\; \sin(m\psi)\; d\psi.$$

In terms of these integrals we have the desired expressions for L, again in Harrington's form:

$$L^{tt} = \left[(\rho'-\rho)\cos\beta' - (z'-z)\sin\beta'\right] T_1 - 2\rho\cos\beta'\, T_2$$

$$L^{t\phi} = (z'-z)\, T_3$$

$$L^{\phi t} = \left[\rho'\sin\beta\cos\beta' - \rho\cos\beta\sin\beta' - (z'-z)\sin\beta\sin\beta'\right] T_3$$

$$L^{\phi\phi} = \left[(\rho'-\rho)\cos\beta - (z'-z)\sin\beta\right] T_1 + 2\,\rho'\cos\beta\, T_2\ .$$

In the simplest application of the method of moments, the unknown functions $K^p(t;m)$ are expanded in pulse functions and the error resulting from their use is forced to zero at discrete points. The pulse functions are each unity over short intervals of t, and zero elsewhere. They are non-overlapping and together cover the whole range of t. The test points lie within these intervals, not necessarily at the mid-points. We are therefore required to evaluate integrals of the type

$$\int_{t_2-\delta_1}^{t_2+\delta_2} L^{pp'}(t_1,t')\ \rho'\ dt'$$

where t_1 is an arbitrary test point and the integral corresponds to a pulse function whose non-vanishing range covers another test point, t_2. (From now on the mode number is not

retained explicitly in the notation). Since the quantities $L^{pp'}$ will be obtained by numerical integration over ψ, it is impossible to carry out the subsequent t-integrals analytically, and hence the integral above is further approximated by the simple expression

$$L^{pp'}(t_1,t_2)\ \rho_2\ \Delta \quad .$$

Here ρ_2 is the value of ρ at t_2 and $\Delta = \delta_1 + \delta_2$. If one wished to use more general test functions, the t-integrals would in any case have to be evaluated numerically, and one would again be led to the consideration of integrals of L alone over short t-segments, as above.

This simple approximation will not suffice when $t_1 = t_2$, because the ψ-integrals defining the components of $L(t,t')$ all fail to converge when t and t' coincide. These singularities are integrable, however, and by considering in detail the behavior of $L(t,t')$ when $t' - t$ is small, we can obtain finite expressions for the integrals desired in the case $t_2 = t_1$.

2. ANALYSIS of the SINGULARITIES.

In the expressions for the components of the dyadic M, given in the previous section, many of the terms vanish when $t = t'$, since this implies that $z = z'$ and $\rho = \rho'$. To determine the behavior of these terms when $t' - t$ is small, we return to the formulas for z and ρ as integrals over t. It is now assumed that the profile of the body is such that the tangent angle, β, is continuous with continuous derivatives, near t, so that the expansion

$$\beta' = \beta + \dot{\beta}\,\tau + \frac{1}{2}\,\ddot{\beta}\,\tau^2 + \ldots$$

is possible, where

$$\tau \equiv t' - t$$

and the dots refer to derivatives with respect to t. The profile may well include corners, but they are modelled as smooth curves with very small radii of curvature. When this series is substituted in the integrals for z and ρ, we obtain the expansions:

$$z' = z + \cos\beta\,\tau - \frac{1}{2}\,\dot{\beta}\sin\beta\,\tau^2 + \ldots$$

$$\rho' = \rho + \sin\beta\,\tau + \frac{1}{2}\,\dot{\beta}\cos\beta\,\tau^2 + \ldots$$

where all unprimed quantities are evaluated at t. From these

formulas and the expansion for β' itself, we obtain

$$(\rho'-\rho)\cos\beta - (z'-z)\sin\beta = \frac{1}{2}\dot{\beta}\,\tau^2 + O(\tau^3)$$

$$(\rho'-\rho)\cos\beta' - (z'-z)\sin\beta' = -\frac{1}{2}\dot{\beta}\,\tau^2 + O(\tau^3)$$

and

$$(\rho'-\rho)^2 + (z'-z)^2 = \tau^2 - \frac{1}{12}(\dot{\beta})^2\,\tau^4 + \ldots$$

Therefore, when $t' - t$ is small, the formulas for the components of L reduce to the following:

$$L^{tt} = (-\frac{1}{2}\dot{\beta}\,\tau^2 + \ldots)\,T_1(\tau) - 2\rho\,(\cos\beta - \dot{\beta}\sin\beta\,\tau + \ldots)\,T_2(\tau)$$

$$L^{t\phi} = (\cos\beta\,\tau + \ldots)\,T_3(\tau)$$

$$L^{\phi t} = (-\dot{\beta}\,\rho\,\tau + \ldots)\,T_3(\tau)$$

$$L^{\phi\phi} = (\frac{1}{2}\dot{\beta}\,\tau^2 + \ldots)\,T_1(\tau) + 2(\rho + \sin\beta\,\tau + \ldots)\cos\beta\,T_2(\tau)\ .$$

The higher order terms, represented by the ellipses in these expressions, will not contribute to the t'-integrals, to the order of accuracy of interest, as will be seen below.

The T-integrals depend on t and t', and hence on τ, through R. The τ-dependence has been recognized in the notation above, but the fixed parameter, t, is suppressed. We make the definitions

$$x^2 \equiv (\rho'-\rho)^2 + (z'-z)^2 = \tau^2 + \ldots$$

$$a^2 = \rho'\rho = \rho\,(\rho + \sin\beta\ \tau + \ldots)$$

and substitute in the formula found earlier for R to obtain

$$R^2 = x^2 + 4a^2 \sin^2(\psi/2).$$

Thus, for $\tau = 0$ and ψ very small, x is zero and R is proportional to ψ, which is the cause of the failure of convergence of the T-integrals at this point.

To deal with these integrals, we recall the definition of G and note that

$$(1-z)\ e^z = (1-z)(1+z + \frac{1}{2} z^2 + \ldots) = 1 - \frac{1}{2} z^2 - \frac{1}{3} z^3 + \ldots$$

We can therefore put

$$\frac{1}{z^3}\ (1-z)e^z = \frac{1}{z^3} - \frac{1}{2z} + H(z)$$

where

$$H(z) \equiv \frac{1}{z^3}\left[(1-z)e^z - 1 + \frac{z^2}{2}\right] = -\frac{1}{3} - \frac{z}{8} + \ldots$$

and express G in the desired form:

$$G(R) = \frac{(ik)^3}{4\pi} \{\frac{1}{(ikR)^3} - \frac{1}{2ikR} + H(ikR)\}$$

or

$$G(R) = \frac{1}{4\pi} [\frac{1}{R^3} + \frac{k^2}{2R} + h(R)] .$$

where h is defined by

$$h(R) = (ik)^3 H(ikR) .$$

The new quantity, h(R), is a bounded function of R with the value $\frac{ik^3}{3}$ at R=0. We substitute this representation of G in the T-integrals to find

$$T_1 = \frac{1}{2\pi} \int_0^\pi [\frac{1}{R^3} + \frac{k^2}{2R} + h(R)] \cos\psi \cos(m\psi)\, d\psi$$

$$T_2 = \frac{1}{2\pi} \int_0^\pi [\frac{1}{R^3} + \frac{k^2}{2R} + h(R)] \sin^2(\psi/2) \cos(m\psi)\, d\psi$$

$$T_3 = \frac{i}{2\pi} \int_0^\pi [\frac{1}{R^3} + \frac{k^2}{2R} + h(R)] \sin\psi \sin(m\psi)\, d\psi .$$

The terms of these integrals involving h are convergent for any value of τ, and the singularities obviously arise from the inverse powers of R.

The next step is the expression of the remaining factors in each of the integrands as the sum of two terms, one of which yields a convergent integral even when multiplied by R^{-3}, while

the other allows the singular portions to be evaluated in closed form. The key to this separation is the possibility of a change of the variable of integration, to a new quantity:

$$u \equiv 2 \sin(\psi/2).$$

Since

$$du = \cos(\psi/2)\, d\psi ,$$

we note that integrals involving a power of R, the factor $\cos(\psi/2)$ and a simple function of $\sin(\psi/2)$ will be integrable in closed form. Starting with T_1, we define the quantity

$$A_1(\psi) \equiv \cos\psi \cos(m\psi) - \cos(\psi/2) \left[1 - \frac{4m^2+3}{2} \sin^2(\psi/2)\right] ,$$

and write

$$\cos\psi \cos(m\psi) = \cos(\psi/2)\left[1 - \frac{4m^2+3}{2} \sin^2(\psi/2)\right] + A_1(\psi) .$$

The $\sin^2(\psi/2)$-term has been supplied and its coefficient so chosen as to make $A_1(\psi)$ proportional to ψ^4 as ψ tends to zero. Then the integral of A_1/R^3 is convergent even at τ = 0, and the divergent portions of T_1 are completely contained in the integral

$$U_1 = \frac{1}{2\pi} \int_0^{\pi} \left(\frac{1}{R^3} + \frac{k^2}{2R}\right) \cos(\psi/2) \left[1 - \frac{4m^2+ 3}{2} \sin^2(\psi/2)\right] d\psi.$$

The new variable is now introduced, and U_1 becomes

$$U_1 = \frac{1}{2\pi} \int_0^2 \left\{ \frac{1}{(a^2u^2+x^2)^{3/2}} + \frac{k^2}{2(a^2u^2+ x^2)^{1/2}} \right\} \left(1 - \frac{4m^2+ 3}{8} u^2\right) du.$$

The term involving u^2/R is convergent when $x=0$, and it is removed to be lumped with the other benign integrals. Since

$$\int_0^2 \frac{u^2 du}{(a^2u^2+x^2)^{3/2}} = \frac{1}{a^2} \int_0^2 \frac{du}{(a^2u^2+x^2)^{1/2}} - \frac{x^2}{a^2} \int_0^2 \frac{du}{(a^2u^2+x^2)^{3/2}},$$

we need only the integrals

$$\int_0^2 \frac{du}{(a^2u^2+x^2)^{1/2}} = \frac{1}{a} \log \left[\frac{2a + (4a^2 +x^2)^{1/2}}{|x|}\right]$$

and

$$\int_0^2 \frac{du}{(a^2u^2 +x^2)^{3/2}} = \frac{2}{x^2(4a^2+x^2)^{1/2}}$$

to complete the evaluation of U_1. Collecting these steps, we have shown that

$$T_1 = \frac{1}{2\pi} \int_0^{\pi} \left[\frac{1}{R^3} + \frac{k^2}{2R} + h(R)\right] \left\{\cos(\psi/2)\left[1-\frac{4m^2+3}{2} \sin^2(\psi/2)\right] + A_1(\psi)\right\} d\psi$$

$$= U_1 + W_1$$

where

$$U_1 = \frac{1}{2\pi} \int_0^2 \left\{ \frac{1 - \frac{4m^2+3}{8} u^2}{(a^2u^2+x^2)^{3/2}} + \frac{k^2/2}{(a^2u^2+x^2)^{1/2}} \right\} du$$

$$= \frac{1}{\pi x^2} \left(1 + \frac{4m^2+3}{8} \frac{x^2}{a^2}\right) (4a^2 + x^2)^{-1/2}$$

$$- \frac{1}{4\pi a^3} \left(\frac{4m^2 + 3}{4} - k^2a^2\right) \log \left\{ \frac{2a + (4a^2+x^2)^{1/2}}{|x|} \right\}$$

and

$$W_1 = 2 \int_0^\pi \left\{ G(R) \cos\psi \cos(m\psi) - \frac{k^2}{8\pi R} \cos(\psi/2) \right.$$

$$\left. - \frac{1}{4\pi R^3} \cos(\psi/2) \left[1 - \frac{4m^2+3}{2} \sin^2(\psi/2)\right] \right\} d\psi.$$

The integral W_1 is convergent at $\tau=0$, and its integral is well-behaved for all τ. For small values of x, we have the expansion

$$U_1 = \frac{1}{2\pi a\ x^2} - \frac{1}{4\pi a^3} \left(\frac{4m^2+3}{4} - k^2a^2\right) \log \left(\frac{4a}{|x|}\right) + \ldots$$

which can be used to assess the diagonal matrix element terms containing T_1. Only the leading term of this expansion will be needed in the end. It should be noted that ρ, and hence a, is

assumed not to be small, so that a thin wire is excluded from our analysis.

The other two integrals are treated in the same way, beginning with the definitions

$$A_2(\psi) \equiv \sin^2(\psi/2)\ [\cos(m\psi) - \cos(\psi/2)]$$

$$A_3(\psi) \equiv \sin\psi\ [\sin(m\psi) - 2m\ \sin(\psi/2)]$$

$$= \sin\psi\ \sin(m\psi) - 4m\ \cos(\psi/2)\ \sin^2(\psi/2) .$$

These A-functions are again $O(\psi^4)$, and T_2 and T_3 are written

$$T_2 = \frac{1}{2\pi} \int_0^\pi \left[\frac{1}{R^3} + \frac{k^2}{2R} + h(R) \right] \{\cos(\psi/2)\ \sin^2(\psi/2) + A_2(\psi)\}\ d\psi$$

and

$$T_3 = \frac{i}{2\pi} \int_0^\pi \left[\frac{1}{R^3} + \frac{k^2}{2R} + h(R) \right] \{4m\ \cos(\psi/2)\ \sin^2(\psi/2) + A_3(\psi)\}\ d\psi$$

The only terms of these integrals which are divergent at $\tau=0$ are

$$U_2 = \frac{1}{2\pi} \int_0^\pi \frac{1}{R^3} \cos(\psi/2)\ \sin^2(\psi/2)\ d\psi$$

$$= \frac{1}{8\pi} \int_0^2 \frac{u^2}{(a^2u^2+x^2)^{3/2}}\ du$$

and

$$U_3 = 4im\, U_2 .$$

This integral has already been evaluated, and to complete the analogy to T_1, we define

$$T_2 = U_2 + W_2$$

and

$$T_3 = U_3 + W_3 ,$$

where

$$U_2 = \frac{1}{8\pi a^3} \log \left\{\frac{2a + (4a^2 + x^2)^{1/2}}{|x|}\right\} - \frac{1}{4\pi a^2} (4a^2 + x^2)^{-1/2}$$

$$U_3 = 4im\, U_2$$

$$W_2 = 2 \int_0^{\pi} \left\{G(R) \sin^2(\psi/2) \cos(m\psi) - \frac{1}{4\pi R^3} \cos(\psi/2) \sin^2(\psi/2)\right\} d\psi$$

and

$$W_3 = 2i \int_0^{\pi} \left\{G(R) \sin\psi \sin(m\psi) - \frac{m}{\pi R^3} \cos(\psi/2) \sin^2(\psi/2)\right\} d\psi .$$

Like W_1, W_2 and W_3 are well-behaved and convergent at $\tau=0$,

and U_2 has the expansion

$$U_2 = \frac{1}{8\pi a^3} \left\{\log \left(\frac{4a}{|x|}\right) - 1\right\} + \ldots$$

3. EVALUATION of the MATRIX ELEMENTS.

As mentioned in Section 1, to apply the method of moments we need to evaluate integrals of the form

$$\int_{t_2-\delta_1}^{t_2+\;{}_2} L^{\rho\rho'}(t_1,t')\ \rho' dt'$$

where t_1 and t_2 are test points of the procedure. When these points are distinct, the simple approximation

$$L^{\rho\rho'}(t_1,t_2)\ \rho_2\ (\delta_1+\delta_2)$$

will suffice, and the T-integrals required are obtained by numerical integration. The original integrands of these integrals will usually be sufficiently well behaved to cause no difficulty here. The mimimum value of R attained in such an integral is just x, and in this case

$$x = [(\rho_2-\rho_1)^2 + (z_2-z_1)^2]^{1/2}$$

is the chord connecting the profile points t_1 and t_2. The decomposition of the T-integrals into the explicitly evaluated U-functions and the well behaved W-integrals would be a useful aid to numerical integration if kx were small, but this is not likely to be the case unless an excessive number of points had been used to describe the body.

Turning to the diagonal matrix elements, we write the required integrals over t' in the form

$$\int_{-\delta_1}^{\delta_2} L^{pp'}(\tau)\ \rho' d\tau$$

where $\tau = t' - t$ as before, and the t-dependence is suppressed in the integrand. In evaluating these integrals we shall retain only terms of the order $\Delta = \delta_1 + \delta_2$ or larger. The formulas of the preceding Section for $L^{pp'}$ are now restated:

$$L^{tt} = -\frac{1}{2}\ \dot{\beta}\ \tau^2\ (U_1+W_1) - 2\rho\ \cos\beta\ (U_2+W_2)$$

$$L^{t\phi} = \cos\beta\ \tau\ (U_3+W_3)$$

$$L^{\phi t} = -\ \dot{\beta}\ \rho\ \tau\ (U_3+W_3)$$

$$L^{\phi\phi} = \frac{1}{2}\ \dot{\beta}\ \tau^2\ (U_1+W_1) + 2\ \rho\ \cos\beta\ (U_2+W_2)\ .$$

In Section 2 the exact formulas for the U-functions were expressed in terms of x, which in the present context is

$$x = [(\rho'-\rho)^2 + (z'-z)^2]^{1/2} = \tau - \frac{1}{24}\ (\dot{\beta})^2\ \tau^3 + \ldots$$

It is easily verified that x may be replaced by τ, and the leading terms of the expansions of these functions used to evaluate the integrals over τ to the required accuracy. Similarly, a may be replaced by ρ in these expressions, which then become simply

$$U_1 = \frac{1}{2\pi\rho\tau^2} + \ldots$$

$$U_2 = \frac{1}{8\pi\rho^3} \left\{\log \left(\frac{4\rho}{|\tau|}\right) - 1\right\} + \ldots$$

and

$$U_3 = 4im\, U_2$$

The W-integrals, of course, are O(1) with regard to τ, and hence the contributions of W_1 and W_3 to the final matrix elements which contain them are of lower order than Δ. Since

$$\int \tau \log \tau \, d\tau = \frac{1}{2} \tau^2 \left(\log \tau - \frac{1}{2}\right) ,$$

even U_3 fails to survive the integration over τ, and thus $L^{t\phi}$ and $L^{\phi t}$ are zero to the desired accuracy.

We are left with

$$L^{tt} = - \frac{1}{2} \dot{\beta} \tau^2 U_1 - 2 \rho \cos\beta \, (U_2+W_2)$$

and

$$L^{\phi\phi} = - L^{tt}.$$

Because of the factor τ^2, only the leading term of U_1 contributes, as already noted, and we can replace ρ' by ρ here to obtain

$$\frac{1}{2}\dot{\beta}\int_{-\delta_1}^{\delta_2} \tau^2 U_1 \ \rho' d\tau$$

$$= \frac{\dot{\beta}}{4\pi}\int_{-\delta_1}^{\delta_2} d\tau = \frac{\Delta\beta}{4\pi} \quad ,$$

where $\Delta\beta = \dot{\beta}\Delta$ is the change in tangent angle of the body profile over the range of the pulse function in question. The significance of this term is discussed in the next Section.

There remains the integral

$$2\rho \cos\beta \int_{-\delta_1}^{\delta_2} [U_2(\tau) + W_2(\tau)] \ \rho' d\tau$$

$$= 2\rho^2 \cos\beta \ W_2(0)\Delta + \frac{\cos\beta}{4\pi\rho}\int_{-\delta_1}^{\delta_2} \{\log (\frac{4\rho}{|\tau|}) - 1\} \ d\tau,$$

in which we have again replaced the factor ρ' by ρ. In addition, $W_2(\tau)$ has been replaced by its value at $\tau = 0$, since higher terms in the series expansion of W_2 would not contribute. We express the logarithmic integral in the following way:

$$\int_{-\delta_1}^{\delta_2} \log(|\tau|)\, d\tau = \delta_2 \log(\delta_2/e) + \delta_1 \log(\delta_1/e)$$

$$= \mu\Delta \log(\frac{\mu\Delta}{e}) + (1-\mu)\Delta \log[\frac{(1-\mu)\Delta}{e}]$$

where μ is defined by the relations

$$\delta_2 = \mu\Delta$$

$$\delta_1 = (1-\mu)\Delta.$$

Then

$$\int_{-\delta_1}^{\delta_2} \log(|\tau|)\, d\tau$$

$$= \Delta \log(\Delta/e) + \Delta [\mu \log\mu + (1-\mu) \log(1-\mu)]$$

$$= \Delta \log(\frac{r\Delta}{e})$$

where

$$r \equiv \mu^{\mu} \cdot (1-\mu)^{(1-\mu)} \quad .$$

The value of the parameter, r, depends on the position of the test point in question within the range of the pulse function which contains it. In general, the profile of the body is defined by a sequence of pairs (z_m, ρ_m), and there is no guarantee that test points will fall at the mid-points of pulse function ranges unless this is deliberately arranged. If μ happens to be one-half, then r is also one-half, and in any case

$$\frac{1}{2} \leq r \leq 1 \quad .$$

Collecting results, we have obtained the following formula for the diagonal matrix element:

$$\int_{-\delta_1}^{\delta_2} L^{tt}(\tau)\ \rho'\ d\tau = -\frac{\Delta\beta}{4\pi}$$

$$+ \frac{\cos\beta}{4\pi\rho}\ \Delta\ [\log(\frac{r\Delta}{4e\rho}) + 1 - 8\pi\rho^3\ W_2(0)] \ .$$

and, of course, $L^{\phi\phi} = - L^{tt}$.

In the integral over ψ which defines $W_2(0)$, we have simply

$$R = 2\rho \sin(\psi/2) .$$

Using the definitions of W_2 and $G(R)$, we can write

$$8\pi\rho^3 W_2(0) = \frac{1}{2} \int_0^{\pi} F(\psi) \frac{d\psi}{\sin(\psi/2)} ,$$

where

$$F(\psi) = (1 - ikR) \exp(ikR) \cos(m\psi) - \cos(\psi/2) .$$

The complete integrand vanishes at $\psi = 0$, and for the purpose of numerical evaluation, the leading term of the expansion

$$\frac{F(\psi)}{\sin(\psi/2)} = (k^2\rho^2 - m^2 + \frac{1}{4}) \psi + \dots$$

may be employed for small ψ.

4. DISCUSSION of the H-FIELD SOLUTION.

Our results for the diagonal matrix elements differ from those of Harrington[2] because of the treatment of the divergent integrals. Harrington avoids divergence of all the T-integrals by moving the test point slightly off the surface, by a distance equal to $\Delta/4$ in our notation. Diagonal matrix elements are then evaluated by the same approximation as off-diagonals, but the integrals are now finite even for $t_1 = t_2$. In the formulas for $L^{pp'}$, the coefficients of all the T-integrals except T_2 vanish in this case, yielding $L^{t\phi} = L^{\phi t} = 0$, but the T_1-term in L^{tt} and $L^{\phi\phi}$ is lost, and the T_2 contributions are evaluated somewhat differently. We defer discussion of the terms $\pm\Delta\beta/4\pi$, which are obtained from T_1, and show first that the two evaluations of the T_2-terms are in fact very similar.

Aside from terms that vanish with τ, we have shown that

$$T_2(\tau) = -\frac{1}{8\pi\rho^3}\log(|\tau|) + K_o$$

where K_o is a constant, evaluated in Section 3. When we integrated over τ, we obtained

$$\int_{-\delta_1}^{\delta_2} T_2(\tau)\, d\tau = -\frac{1}{8\pi\rho^3}\,\Delta \log\left(\frac{r\Delta}{e}\right) + K_o\Delta$$

which, in turn, is equal to

$$\Delta\, T_2\left(\frac{r\Delta}{e}\right)$$

to this accuracy. If we had placed the field point, r, off the surface by a distance ζ along the positive normal, we would have found

$$R = [x^2 + 4a^2 \sin^2(\psi/2)]^{1/2}$$

as before, but where x and a now have the more general definitions

$$x^2 = (\rho'-\rho)^2 + (z'-z)^2$$
$$-2\zeta[(\rho'-\rho)\cos\beta - (z'-z)\sin\beta] + \zeta^2$$

and

$$a^2 = \rho'(\rho+\zeta\cos\beta)\ .$$

If now, we let $t = t'$, we obtain simply

$$x^2 = \zeta^2$$

$$a^2 = \rho(\rho + \zeta \cos\beta) \quad ,$$

and since ζ is small, the asymptotic expansion derived in Section 3 is valid for T_2, but now in terms of x. Again neglecting terms which vanish with ζ, the value of T_2, with $\tau = 0$ and ζ small, is the same as the expression above, but with ζ replacing τ. Consequently, if ζ is chosen equal to $r\Delta/e$, and the T_2 integral evaluated numerically, this procedure should give the same results as our formulas, to the desired accuracy. Recalling the range of r, we see that

$$\frac{1}{2e} \leq \frac{r}{e} \leq \frac{1}{e}$$

which covers the value 1/4 used by Harrington.

Regarding the new term, $\pm\Delta\beta/4\pi$, we combine it with the diagonal contribution of 1/2 made by the explicit term

$$\frac{1}{2} K^p(t;m)$$

in the modal integral equation. This combination is then

$$\frac{1}{2}\left(1 - \frac{\Delta\beta}{2\pi}\right) K^t$$

in the t-equation, and

$$\frac{1}{2}\left(1 + \frac{\Delta\beta}{2\pi}\right) K^\phi$$

in the ϕ-equation. For a convex portion of the body profile, $\Delta\beta$ is negative by our sign conventions. For a smooth surface and a reasonable density of points, $\Delta\beta/2\pi$ will be small compared to unity, and the new term will have little effect. It may, however, be comparable to the remaining part of L^{tt} or $L^{\phi\phi}$. The comparison here is between $\Delta\beta$ and

$$\cos\beta \, \frac{\Delta}{\rho} \left[\log\left(\frac{\Delta}{\rho}\right) + K'\right] \quad ,$$

where

$$K' = 1 + \log\left(\frac{r}{4e}\right) - 8\pi\rho^3 \, W_2(0) \ .$$

The ratio Δ/ρ will generally be small (thin wires having been excluded), and the expression

$$\left|\frac{\Delta}{\rho} \log\left(\frac{\Delta}{\rho}\right)\right|$$

can not exceed $1/e$, for values of Δ less than ρ. Thus $\Delta\beta$ should be compared to the bound

$$\cos\beta \left(\frac{1}{e} + K' \frac{\Delta}{\rho}\right)$$

for the remaining terms of this matrix element.

The most likely situation in which the new term could have an effect is in the immediate vicinity of a corner of the profile, as at the base of a cone. In order to accomodate this term, the scheme for describing the body must allow for the pre-processing of a sequence of profile points to yield pulse function boundaries, test points and values of $\dot{\beta}$ or $\Delta\beta$ at the latter. If a pulse function spans a corner, the corresponding $\Delta\beta$ should equal or closely approximate the actual change in tangent direction at this point. In this case, the total diagonal matrix elements can be significantly altered.

The combination of the $\Delta\beta$ terms with the existing 1/2 in the diagonals yields expressions which are very similar to the formulas given by Poggio and Miller[3] for the H-field integral equation, when the field point falls at a corner. In our notation, their term is

$$\frac{1}{2}\left(1 - \frac{\Delta\beta}{2\pi}\right) K^P$$

which applies to either value of p. We assumed that the surface was actually smooth, although the radius of curvature at a 'corner' of the profile could be extremely small compared to the wavelength. Since the limiting process implicit in the derivation of the Maue equation involves taking the field point right up to the surface, there is no natural scale against which non-zero radii of curvature may be compared. In the case of the formula of Poggio and Miller, a mathematically precise corner is assumed, and their modified term applies only when the field point falls at this corner. For the t-equation, our formulation will give the same result as theirs, but the formulas differ for the ϕ-equation. This is less disturbing than it might be, due to the fact that the ϕ-component of surface current is (integrably) infinite at a mathematically precise corner. Maue[1] makes the case for this statement in a general context, and he obtains the same behavior for the component of current along an edge, such as the junction of a cone with a flat base, as is exhibited by the exact solution for a wedge.

Returning to the first equation given in Section 1, we know from potential theory that the tangential component of the surface integral, as a function of r, is discontinuous at the surface, suffering a jump equal to K(r) at this point. We also know that when r is placed on the surface, this integral is

convergent, in the sense that the limit approached, when a small region covering r is exluded from the integral, is independent of the shape of this region, as its maximum chord goes to zero. Maue's equaton results from the fact that the value to which this integral converges is the average of its values when approached from opposite sides of the surface. If we did not know this, we could still obtain an integral equaton for K by actually taking the limit of the integral

$$\underline{n} \times \int_S K(r') \times \nabla' g(r,r')\, dS'$$

as r approached S from the outside. As an exercise, this procedure is carried out in Appendix 2 for the individual modal integral equations with the result, of course, that the term $K^p(t;m)$ is replaced by $1/2\ K^p(t;m)$, and the remaining integrals all converge. The validity of this derivation depends entirely on the integral T_1 and the asymptotic expansion derived for it in Section 3. The general technique may be useful in other situatons, such as the form of the E-field equation discussed in Section 5, where a similar surface integral is involved which can be evaluated in finite form as a limit (as r approaches S), but in which one cannot put r actually on S and still have a convergent integral.

5. THE E-FIELD SOLUTION.

If E_o is a source field, and r is a point exterior to the surface S of a perfect conductor, then the electric field satisfies the equation

$$E(r) = E_o(r) + i\mu\omega \int_S \left\{ K(r')\, g(r,r') + \frac{1}{k^2} [K(r')\cdot\nabla'] \nabla' g(r,r') \right\} dS'$$

where K is again $\underline{n} \times H$ on the surface and μ is the magnetic permeability. This form, of course, follows directly from the use of the standard dyadic Green's function[4], and the second term of the integrand is usually transformed as follows:

$$\int_S [K(r') \cdot \nabla'] \nabla' g(r,r')\, dS'$$

$$= - \nabla \int_S K(r') \cdot \nabla' g(r,r')\, dS'$$

$$= \nabla \int g(r,r')\, \nabla' \cdot K(r')\, dS'.$$

The field point is now allowed to approach the surface, and the boundary condition $\underline{n} \times E = 0$ is applied, with the result

$$\frac{i}{\mu\omega} \underline{n} \times E_o(r) = \lim_{r \to S} \underline{n} \times \int_S g(r,r')\, K(r')\, dS'$$

$$+ \frac{1}{k^2} \lim_{r \to S} (\underline{n} \times \nabla) \int_S g(r,r')\, \nabla' \cdot K(r')\, dS' \quad .$$

From potential theory we know that this surface integral and its tangential derivatives are continuous functions of r, on or off the surface, hence the limits are taken by simply placing r on S, and no convergence difficulty arises.

The method of moments is now applied[2], with expansion and test functions in the Galerkin procedure. The derivative of the expansion function, inside the integral over r', is approximated by a difference operation, and the ∇ outside the integral is transferred to the test function upon integration over r, where it is also approximated by a difference.

When carried out in detail, the following ψ-integrals occur:

$$S_1 = 2 \int_0^{\pi} g(R) \cos(m\psi)\, d\psi$$

$$S_2 = 2 \int_0^{\pi} g(R) \cos\psi \cos(m\psi)\, d\psi$$

and

$$S_3 = 2i \int_0^{\pi} g(R) \sin\psi \sin(m\psi)\, d\psi \quad .$$

The leading term in the expansion of $g(R)$ for small R is simply $1/4\pi R$, hence S_3 is convergent and the others are simple. The divergent portion of S_1 and S_2 is just

$$\frac{1}{2\pi} \int_0^{\pi} \frac{\cos(\psi/2)}{R} \, d\psi$$

$$= \frac{1}{2\pi} \int_0^{2} \frac{du}{(a^2u^2 + x^2)^{1/2}} \quad .$$

This integral is logarithmic and it can be dealt with just as T_2 was in Section 3. In particular, the field point can be removed a distance $r\Delta/e$ from the surface and all terms become finite. No contribution involving $\Delta\beta$ arises. This is consistent with the E-field equation of Poggio and Miller, since the term in their equation affected by the presence of a corner vanishes when the boundary condition for a conductor is applied.

Another E-field equation can be obtained from the first equation of this section by letting r approach the surface and then equating the tangential components of E to zero:

$$\frac{i}{\mu\omega} \underline{n} \times E_o(r) = \lim_{r \to S} \underline{n} \times \int_S \{K(r)\, g(r,r')$$

$$+ \frac{1}{k^2} [K(r') \cdot \nabla'] \, \nabla' g\,(r,r')\} \, dS' \quad .$$

The differential operators are now treated exactly, since the derivatives are carried out explicitly on the scalar Green's function, at the cost of a more complicated kernel for the integral equation (see Andreasen[5]). The integral will be convergent only if r approaches S after the integrations are performed.

The methods of this report have been applied to this equation and it is found that all integrals remain finite, but the analysis is much more complicated in detail. Nine ψ-integrals occur, many with inverse fifth powers of R, and the asymptotic forms are more tedious to obtain. A particular feature arises which may well cause difficulty in a numerical analysis based on these equations, however. One of the diagonal matrix element terms turns out to be inversely proportional to Δ, the pulse function range in the t-direction. If our surface integral is used to find the electric field at a point close to a plane surface, due to a small region near the point, an expression is obtained which is inversely proportional to the linear dimensions of that region, as the point approaches the surface. The formula obtained is strongly dependent on the shape of the region, and in our case is proportional to $1/\Delta$. The problem is not one of convergence, since Δ is not infinitesimal, but it may lead to numerical instability, thus defeating the purpose of this scheme,

which is to avoid the effects of approximating derivatives by differences. A report of this method will have to await the results of trial by actual numerical analysis.

APPENDIX 1. EVALUATION of the DYADIC COMPONENTS.

Let $\underline{z}$, $\underline{\rho}$ and $\underline{\phi}$ be unit vectors in a cylindrical coordinate system. The directions of $\underline{\rho}$ and $\underline{\phi}$ depend, of course, on location, and for two points on the surface of the body, the relations

$$\underline{\rho} \cdot \underline{\rho}' = \underline{\phi} \cdot \underline{\phi}' = \cos\psi$$

and

$$\underline{\phi} \cdot \underline{\rho}' = - \underline{\rho} \cdot \underline{\phi}' = \sin\psi$$

are valid, where $\psi = \phi' - \phi$. The normal, $\underline{n}$, and the tangent unit vector, $\underline{t}$, are expressed in terms of $\underline{z}$ and $\underline{\rho}$ by the equations

$$\underline{t} = \cos\beta\ \underline{z} + \sin\beta\ \underline{\rho} \quad , \underline{n} = -\sin\beta\ \underline{z} + \cos\beta\ \underline{\rho} \quad .$$

If the field point, r, is off the surface, in the direction of $\underline{n}$, by a small amount ζ, then we can write

$$r = z\ \underline{z} + \rho\ \underline{\rho} + \zeta\underline{n} = (z - \zeta \sin\beta)\ \underline{z} + (\rho + \zeta \cos\beta)\ \underline{\rho} \ .$$

Working out the cross-products such as $\underline{z} \times \underline{\rho} = \phi$, we find that

$$\underline{t} \times r = (\rho \cos\beta - z \sin\beta + \zeta)\ \underline{\phi}$$

and

$$\underline{\phi} \times r = (z - \zeta \sin\beta)\ \underline{\rho} - (\rho + \zeta \cos\beta)\ \underline{z} \quad .$$

At r', the point is on the surface, hence the analogous

quantities are

$$\underline{t}' \times r' = (\rho' \cos\beta' - z' \sin\beta')\ \underline{\phi}'$$

and

$$\underline{\phi}' \times r' = z'\underline{\rho}' - \rho'\ \underline{z} \quad .$$

The dot products required for the components of M are now evaluated:

$$(\underline{t} \times r) \cdot \underline{t}' = (\rho \cos\beta - z \sin\beta + \zeta) \sin\beta' \sin\psi$$

$$(t \times r) \cdot \underline{\phi}' = (\rho \cos\beta - z \sin\beta + \zeta) \cos\psi$$

$$(\underline{\phi} \times r) \cdot \underline{t}' = -\ (\rho + \zeta \cos\beta) \cos\beta' + (z - \zeta \sin\beta) \sin\beta' \cos\psi$$

$$(\underline{\phi} \times r) \cdot \underline{\phi}' = -\ (z - \zeta \sin\beta) \sin\psi \quad .$$

The corresponding products

$$(\underline{p}' \times r') \cdot \underline{q}$$

are obtained by interchanging primed and unprimed quantities and omitting terms proportional to ζ.

These results are combined, and the identity

$$1 = \cos\psi + 2 \sin^2(\psi/2)$$

is used, where appropriate, to obtain the desired formulas

$$\frac{M^{tt}}{G(R)} = (\underline{\phi} \times r) \cdot \underline{t}' + \underline{\phi} \cdot (\underline{t}' \times r')$$

$$= [(\rho'-\rho) \cos\beta' - (z'-z) \sin\beta'] \cos\psi - \zeta \cos(\beta'-\beta) \cos\psi$$

$$-\ 2\ (\rho + \zeta \cos\beta) \cos\beta' \sin^2(\psi/2)$$

$$\frac{M^{t\phi}}{G(R)} = (\underline{\phi}\times r) \cdot \underline{\phi}' + \underline{\phi} \cdot (\underline{\phi}'\times r')$$

$$= (z' - z + \zeta \sin\beta) \sin\psi$$

$$\frac{M^{\phi t}}{G(R)} = - (\underline{t}\times r) \cdot \underline{t}' - \underline{t} \cdot (\underline{t}'\times r')$$

$$= [\rho' \sin\beta \cos\beta' - \rho \cos\beta \sin\beta' - (z'-z) \sin\beta \sin\beta'$$

$$- \zeta \sin\beta'] \sin\psi$$

$$\frac{M^{\phi\phi}}{G(R)} = - (\underline{t}\times r) \cdot \underline{\phi}' - \underline{t} \cdot (\underline{\phi}'\times r')$$

$$= [(\rho'-\rho) \cos\beta - (z'-z) \sin\beta - \zeta] \cos\psi$$

$$+ 2 \rho' \cos\beta \sin^2(\psi/2) \quad .$$

APPENDIX 2. DERIVATION of the MODAL EQUATIONS as LIMITS.

If the analysis of the main report is repeated with the field point, r, off the surface by a small distance ζ in the direction of the positive normal, the modal equations will be

$$K_o^p(t;m) = K^p(t;m) + \lim_{\zeta \to o} \sum_{p'} \int L^{pp'}(t,t';m)\, K^{p'}(t';m)\, \rho' dt' \ .$$

Using the results of Appendix 1, one finds that the dyadic components are now given by

$$L^{tt} = [(\rho'-\rho)\cos\beta' - (z'-z)\sin\beta' - \zeta\cos(\beta'-\beta)]\, T_1$$
$$- 2(\rho+\zeta\cos\beta)\cos\beta'\, T_2$$

$$L^{t\phi} = (z'-z+\zeta\sin\beta)\, T_3$$

$$L^{\phi t} = [\rho'\sin\beta\cos\beta' - \rho\cos\beta\sin\beta' - (z'-z)\sin\beta\sin\beta'$$
$$- \zeta\sin\beta']\, T_3$$

and

$$L^{\phi\phi} = [(\rho'-\rho)\cos\beta - (z'-z)\sin\beta - \zeta]\, T_1$$
$$+ 2\rho'\cos\beta\, T_2 \ .$$

The T-integrals have the same definitions as before, but as noted in Section 4, we now have

$$R^2 = x^2 + 4a^2 \sin^2(\psi/2) ,$$

where

$$x^2 = (\rho'-\rho)^2 + (z'-z)^2$$
$$- 2 \zeta [(\rho'-\rho) \cos\beta - (z'-z) \sin\beta] + \zeta^2$$

and

$$a^2 = \rho' (\rho+\zeta \cos\beta) .$$

The entire discussion of these integrals is still applicable, including the asymptotic expansions for small values of x.

We can simply put $\zeta = 0$ everywhere, except in the vicinity of the point $t' = t$, where we must evaluate the effect of the ζ-terms. In those terms of the dyadic components which are identical to terms already discussed (with r on the surface), we can immediately put $\zeta = 0$ for any t', since these terms have all been shown to be finite after integration over t'. Similarly, in the terms ζT_2 and ζT_3, the T-factors yield convergent integrals over t' (T_3 behaves just like T_2), hence these terms will vanish when ζ goes to zero.

The only terms which remain are

$$\Delta L^{tt} = - \zeta \cos(\beta' - \beta)\ T_1$$

and

$$\Delta L^{\phi\phi} = - \zeta\ T_1 \quad .$$

If we expand $\cos(\beta' - \beta)$, the second term is proportional to τ^2, and we know that $\tau^2 T_1$ is integrable, so that in the limit, we need retain only

$$\Delta L^{tt} = \Delta L^{\phi\phi} = - \zeta\ T_1 \quad .$$

For T_1 we can take the first term of its expansion:

$$T_1 = \frac{1}{2\pi a x^2}$$

since all others are finite after integration over t'. The contribution of the new terms thus reduces to

$$\lim_{\zeta \to 0} \int_{-\delta_1}^{\delta_2} \Delta\ L^{tt}(\tau)\ K^t(t')\ \rho'\ d\tau$$

$$= -\frac{K^t(t)}{2\pi} \lim_{\zeta \to 0} \int_{-\delta_1}^{\delta_2} \frac{\rho' d\tau}{a\, x^2} ,$$

with an exactly analogous expression for the ϕ-term. In the present context, we may think of δ_1 and δ_2 as infinitesimal.

Using the expansions derived in Section 2, we have

$$\frac{\rho'}{a} = \left(\frac{\rho'}{\rho + \zeta \cos\beta}\right)^{1/2} = 1 + \frac{1}{2\rho} (\sin\beta\ \tau - \cos\beta\ \zeta) + \ldots$$

and

$$x^2 = (1 - \dot{\beta}\ \zeta)\ \tau^2 + \zeta^2 + \ldots$$

where, as usual, terms of higher order than those explictly indicated turn out to make no contribution in the limit. We define

$$\sigma^2 \equiv \frac{\zeta^2}{1 - \dot{\beta}\ \zeta}$$

and substitute:

$$\lim_{\zeta \to 0} \zeta \int_{-\delta_1}^{\delta_2} \frac{\rho'\, d\tau}{a\, x^2}$$

$$= \lim_{\zeta \to 0} \frac{\zeta}{1 - \dot{\beta}\,\zeta} \int_{-\delta_1}^{\delta_2} \left(1 - \frac{\cos\beta}{2\rho}\,\zeta + \frac{\sin\beta}{2\rho}\,\tau\right) \frac{d\tau}{\tau^2 + \sigma^2} \quad .$$

Since

$$\int_{-\delta_1}^{\delta_2} \frac{\tau\, d\tau}{\tau^2 + \sigma^2} = \frac{1}{2} \log \left(\frac{\delta_2^2 + \sigma^2}{\delta_1^2 + \sigma^2}\right)$$

is finite for $\sigma = 0$, the corresponding term, when multiplied by ζ, will vanish in the limit. We are left with

$$\lim_{\zeta \to 0} \frac{\zeta}{1 - \dot{\beta}\,\zeta} \left(1 - \frac{\cos\beta}{2\rho}\,\zeta\right) \int_{-\delta_1}^{\delta_2} \frac{d\tau}{\tau^2 + \sigma^2}$$

$$= \lim_{\sigma \to 0} \sigma \int_{-\delta_1}^{\delta_2} \frac{d\tau}{\tau^2 + \sigma^2}$$

$$= \lim_{\sigma \to 0} \left\{\tan^{-1}\left(\frac{\delta_2}{\sigma}\right) + \tan^{-1}\left(\frac{\delta_1}{\sigma}\right)\right\}$$

$$= \pi \quad .$$

If ζ had approached the surface from the inside, σ would have passed through negative values to zero, and the result would have been $-\pi$. This accounts for the discontinuity of the surface integral. Altogether, we have found

$$\lim_{\zeta \to 0} \int_{-\delta_1}^{\delta_2} \Delta L^{tt}(\tau)\, K^t(t')\, \rho'\, d\tau = -\frac{1}{2} K^t(t) \quad ,$$

with the same result for the integral of $\Delta L^{\phi\phi}$. The modal integral equations then revert to the forms derived from the Maue equation directly.

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The basic formulation of scattering problems in terms of integral equations is examined, for the special case of perfectly conducting bodies of revolution. In particular, the singularities of the integrals which arise in this context are studied, in relation to the transition to a numerical solution by means of the method of moments. For both H-field and E-field equations, it is found that finite matrix elements can be deduced in a way that is uniquely determined by the integrals themselves. No ad hoc procedures are required to secure convergence, but one such procedure, which is commonly used, is shown to be capable of giving accurate results for those integrals which tend to diverge logarithmically. In the H-field solution, other integrals also arise, which yield finite terms not normally included in the theory, when the necessary limiting procedures are carried out. These terms can play a significant role in the case of a body whose profile contains a corner.

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